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Study of a Class of Models for Self-Organization: Equilibrium Analysis[1]

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A new class of nonlinear stochastic models is introduced with a view to explore self-organization. The model consists of an assembly of anharmonic oscillators, interacting via a mean field of system size range, in presence of white, Gaussian noise. Its properties are explored in the overdamped regime (Smoluchowski limit). The single oscillator potential is such that for small oscillator displacements it leads to a highly nonlinear force but becomes asymptotically harmonic. The shape of the potential can be a single- or double-well and is controlled by a set of parameters. Through equilibrium statistical mechanical analysis, we study the collective behavior and the nature of phase transition. Much of the analysis is analytic and exact. The treatment is not restricted to the thermodynamic limit so that we are also able to discuss finite size effects in the model.

KEY WORDS: Self-organization; nonlinearity; fluctuations and noise; double-well potentials; order-parameter; phase transitions; system-size effects.

You boil it in sawdust, you salt it in glue,
You condense it with locusts and tape,
Still keeping one principal object in view,
To preserve its symmetrical shape.

Lewis Carroll in *The Hunting of the Snark*.

1. INTRODUCTION

In this paper, we use a class of nonlinear stochastic models to study self-organization processes. Often such processes occur in complex systems made up of assemblies of interacting entities each of which may be

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relatively simple. Examples of self-organized behavior abound in nature and the statistical mechanics dealing with such synergetic behavior has found applications[1] not only in physical and biological sciences but also in socioeconomic areas.

Even though many self-organizing systems display a great complexity, the essential ingredients for the process are the coupling between the members of the assembly, feedback processes, and the presence of fluctuations. As a consequence, it is appropriate to consider nonlinear stochastic models[2] as idealized versions of such systems. An appropriate mathematical framework could be a coupled set of nonlinear stochastic differential equations whose solutions are described by joint probability densities. The self-organized behavior occurring on a macroscopic scale will then be obtained via the marginal (reduced) probability densities of the relevant macroscopic variables (order parameter).

The general framework described above has been used earlier[3] to study equilibrium and nonequilibrium properties of a model system of coupled, anharmonic oscillators in Smoluchowski limit and where the anharmonicity originated from a quartic-type potential. In this paper, we consider a qualitatively different class of nonlinear stochastic models and restrict ourselves to explore only its equilibrium properties. As in Ref. 3, our model system is a coupled set of nonlinear oscillators, interacting via a mean field whose range extends to the size of the system, and in presence of a white, Gaussian noise; further, we also restrict ourselves to the overdamped regime. The major difference from Ref. 3 is in the nature of nonlinearity: the oscillator potential in our case is asymptotically harmonic, and depending on the value of the model parameters, can have single- or double-well characteristics. Thus the class of potentials studied in this paper can be viewed as perturbations (not necessarily small) of the harmonic potential. We expect our class of models to be useful in those physical and chemical situations where for example a periodic lattice environment is locally distorted to create a double well situation.[4]

The class of models we study are described in detail in Section 2. In particular the single oscillator potential, given explicitly in Eq. (17), is introduced via Weber's parabolic cylinder functions; these have many interesting mathematical properties[5] which facilitate the calculations of several highly nonlinear problems.[6] One of these properties that we shall implicitly use throughout the paper is that the Weber function possesses a self-similar Fourier transform—a property shared by Gaussians and which permits one to obtain exact solutions in various linear problems.

The rest of the paper is organized as follows. In Section 2, we begin with a dynamical statement of the general problem in terms of the multivariate Fokker–Planck equation for the conditional probability density. Its stationary solution is used to obtain the marginal probability density $P_s(x)$ [see Eq. (9)] which involves the oscillator potential. The curvature at origin of $P_s(x)$ can be used to determine whether the system is in ordered or disordered phase. Through a lemma we obtain a general expression for the curvature at origin which facilitates its evaluation for specific models. The class of model potentials is introduced in Eq. (17) and its general properties discussed. The general form of the potential has four parameters out of which two can be scaled away. Thus after a scaling transformation, the model involves two potential parameters p and q, an interaction parameter θ, and the strength of the noise intensity D. The parameter p is related to the coefficient of the asymptotically harmonic form of the potential and q is a measure of the perturbation (not necessarily small) of the potential locally for small oscillator displacements. The model is formally exactly solved for $D = 1$; for $D \simeq 1$, an excellent approximation Eq. (27) enables us to obtain $P_s(x)$ analytically. This result is given in Eq. (31) and its general features are discussed in Section 3. The formally exact solution in Eq. (31) is valid for any number of oscillators N and we discuss in Section 3 the two extremes of $N \to 1$ and $N \to \infty$; the discussion also includes the behavior of $P_s(x)$ when the interaction parameter θ becomes large. In order to obtain the phase diagram for the system in its parameter space, one needs (in the $N \to \infty$ limit) to know only the curvature (at origin) R of $P_s(x)$. For various cases this is evaluated and phase diagrams that result are shown in Figs. 1 and 2; these diagrams are generic for the entire class of models. In Section 4, we consider system size effects and evaluate in detail two special cases of the general potential. In Section 5, we consider the evaluation of R and the phase diagrams for arbitrary noise intensity D. The use of the lemma proved in Section 2 enables us to evaluate R without making use of the approximation, Eq. (27). We conclude with some further discussion of the results in Section 6. Many of the mathematical details are given in the appendixes.

2. THE CLASS OF MODELS

Our class of models is introduced in the same manner as in Ref. 3. We mention here only briefly the connection to dynamics since this paper deals only with the static properties of these models.

The models are characterized by the set of stochastic differential equations (Langevin equations)

$$dz_j = -\left[\frac{d}{dz_j} V(\mathbf{z})\right] dt + \sqrt{D}\, d\beta_{j,t}, \qquad j = 1 \ldots N, \quad \mathbf{z} \in \mathbb{R}^N \tag{1}$$

where $d\beta_{j,t}$ models a vectorial Gaussian white noise for which we shall further assume

$$\langle d\beta_{j,t} \rangle = 0 \tag{2a}$$

and

$$\langle d\beta_{j,t} d\beta_{k,\tau} \rangle = 2\delta_{jk}\delta(|t-\tau|) \tag{2b}$$

Owing to the presence of white noise, the stochastic process $\mathbf{z}(t)$ is Markovian and its conditional probability density $P(\mathbf{z}, t \mid \mathbf{z}_0, t_0)$ obeys the Fokker–Planck equation

$$\begin{aligned}\frac{\partial}{\partial t} & P(\mathbf{z}, t \mid \mathbf{z}_0, t_0) \\ &= \sum_{j=1}^{N} \frac{\partial}{\partial z_j}\left\{\left[\frac{\partial}{\partial z_j} V(\mathbf{z})\right] P(\mathbf{z}, t \mid \mathbf{z}_0, t_0) + D\frac{\partial}{\partial z_j} P(\mathbf{z}, t \mid \mathbf{z}_0, t_0)\right\}\end{aligned} \tag{3}$$

The dynamics modeled by Eq. (1) being of gradient type with diagonal diffusion tensor [Eq. (2b)], the stationary solution of Eq. (3) obeys detailed balance and then reads

$$P_S(\mathbf{z}) = Q^{-1}\exp\left[-D^{-1}V(\mathbf{z})\right] \tag{4}$$

where Q is the normalization factor.

Following Ref. 3, we take the mean of the coordinates z_j as a relevant dynamical variable and identify its average over the ensemble Eq. (4) with an order parameter x. The marginal probability density for x then reads

$$P_S(x) = \int_{\mathbb{R}^N} \delta\left[\sum_{j=1}^{N}\left(z_j N^{-1}\right) - x\right] P_S(\mathbf{z})\, d^N z \tag{5}$$

or equivalently

$$\begin{aligned}P_S(x) &= (2\pi Q)^{-1} N \int_{\mathbb{R}} d\xi \exp(-i\xi N x) \\ &\quad \times \int_{\mathbb{R}^N} d^N z \exp\left[i\xi \sum_{j=1}^{N} z_j - D^{-1} V(\mathbf{z})\right]\end{aligned} \tag{6}$$

Let us now split the potential $V(\mathbf{z})$ into two parts, namely,

$$\begin{aligned}V(\mathbf{z}) &= \sum_{j=1}^{N} V_S(z_j) + \frac{\theta}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}(z_i - z_j)^2 \\ &= V_1(\mathbf{z}) - \frac{\theta}{2N}\left(\sum_{j=1}^{N} z_j\right)^2, \qquad \theta \in \mathbb{R}^+\end{aligned} \tag{7}$$

where $V_1(\mathbf{z})$ denotes an effective single-particle potential:

$$V_1(\mathbf{z}) = \sum_{j=1}^{N} \left[V_S(z_j) + \frac{\theta}{2} z_j^2 \right] \tag{8}$$

With the definitions Eqs. (7) and (8), it is easy to realize that the stochastic differential equation, Eq. (1), describes an assembly of N statistically independent oscillators with a system-size range of coupling and in the Schmoluchowski limit. Each oscillator has the potential energy $V_s(z)$.

Introducing Eq. (7) into Eq. (6), we find

$$P_S(x) = (2\pi Q)^{-1} N \exp\left(\frac{\theta N x^2}{2D} \right) \int_{\mathbb{R}} d\xi \exp\left[-i\xi N x + N\psi(i\xi) \right] \tag{9}$$

where

$$\exp\left[\psi(i\xi) \right] = \int_{\mathbb{R}} dz \exp\left\{ -D^{-1} \left[V_S(z) + \frac{\theta}{2} z^2 \right] + i\xi z \right\} \tag{9a}$$

In the following, we shall be interested in situations for which $V_s(z)$ may, according to external parameters, exhibit single- versus double-well structure. In particular, for a well-separated and infinitely deep double well, the model Eq. (7) approaches the well-known Curie–Weiss model. In Ref. 3, the authors considered the case $V_s(z) = az^2 + bz^4$ which led to a complex expression for Eq. (9a). In this paper, we shall consider another type of double-well structure $V_s(z)$ which possesses simpler Fourier transforms which arise in Eqs. (9) and (9a).

Before we introduce explicitly our class of potential $V_s(z)$, we shall make the following general remarks:

Let us from now on assume that we deal with symmetric potentials $V_s(z)$. Hence, the marginal density $P_s(x)$ will be itself symmetric and could have either an odd or an even number of maxima. In the unimodal (bimodal) situation, we shall speak of a disordered (ordered) phase. In this paper we do not consider the possibility of $P_s(x)$ having more than two maxima. It is then obvious that the curvature of $P_s(x)$ at the origin completely characterizes the phase to which the system belongs. While for finite N, the curvature of $P_s(x)$ has to be obtained directly from Eqs. (9) and (9a), in the thermodynamic limit ($N \to \infty$), the following lemma holds and helps to ascertain the curvature:

Lemma. Assume $V_s(z) = V_s(-z)$ and such that $P_s(x)$ as given by Eqs. (9) and (9a) exists. Then we have

$$R = \lim_{N \to \infty} N^{-1} \frac{d^2}{dx^2} P_s(x) \bigg|_{x=0} = \text{const} \left\{ \frac{\theta}{D} - \left[\frac{d^2}{d\omega^2} \psi(\omega) \bigg|_{\omega=0} \right]^{-1} \right\} \tag{10}$$

where

$$\omega = i\xi \tag{10a}$$

Proof. As $N \to \infty$, we can apply the steepest descent method to calculate Eq. (9). We then have

$$\lim_{N\to\infty} P_s(x) = \exp\left\{ \ln(\text{const}) + \frac{\theta N x^2}{2D} - \frac{1}{2}\ln\left[\frac{d^2}{d\omega^2}\psi(\omega)\Big|_{\omega=\bar{\omega}} \right] - N\bar{\omega}x + N\psi(\bar{\omega}) \right\} \tag{11}$$

where

$$x = \frac{d}{d\omega}\psi(\omega)\Big|_{\omega=\bar{\omega}} \tag{12}$$

Because $V_s(z) = V_s(-z)$ and hence $P_s(x) = P_s(-x)$ we have

$$\bar{\omega}(x) = \bar{\omega}(-x) \Rightarrow \bar{\omega}(0) = 0 \tag{13}$$

Using Eq. (13), we have

$$\lim_{N\to\infty} N^{-1}\frac{d^2}{dx^2}P_s(x)\Big|_{x=0} = \text{const}\left[\frac{\theta}{D} - 2\frac{d}{dx}\bar{\omega}(x)\Big|_{x=0} + \left[\frac{d^2}{d\omega^2}\psi(\omega)\Big|_{\omega=0} \right]\left[\frac{d}{dx}\bar{\omega}(x)\Big|_{x=0} \right]^2 \right] \tag{14}$$

The Fourier transform of a real symmetric function being itself symmetric, Eq. (12) can be written in the form

$$x = \sum_{\lambda=0}^{\infty} B_\lambda(\bar{\omega})^{2\lambda+1} = \frac{d\psi(\omega)}{d\omega}\Big|_{\omega=\bar{\omega}} \tag{15}$$

and hence

$$\frac{d^2}{d\omega^2}\psi(\omega)\Big|_{\omega=0} = B_0 = \left(\frac{d\bar{\omega}}{dx}\Big|_{x=0} \right)^{-1} \tag{16}$$

The assertion, Eq. (10), follows from Eqs. (16) and (14). ■

In this paper, we shall discuss the results which can be obtained when one considers the class of potentials defined by

$$V_s(z_j) = \frac{p}{2}z_j^2 - \epsilon \ln\left[y_1\left(\frac{q}{2s}, (2s)^{1/2} z_j \right) \right] \tag{17}$$

where the parameters obey

$$p, s, \epsilon, \text{ and } \theta \in \mathbb{R}^+ \tag{17a}$$

and the function $y_1(q/2s,(2s)^{1/2}z)$ is an even solution of the Weber equation[7]:

$$\frac{d^2}{dz^2} y_1\left(\frac{q}{2s},(2s)^{1/2}z\right) = (s^2z^2+q)y_1\left(\frac{q}{2s},(2s)^{1/2}z\right) \tag{18}$$

The range of the parameter q is given below.

Let us remark that Eq. (18) is merely a Schrödinger equation for a single harmonic oscillator. While its (quantum mechanically speaking) physical solution is well known in terms of Hermite polynomials, the other (non-square-integrable) solutions are also useful in constructing a class of potentials which are asymptotically harmonic with strong local perturbations.[6] As we shall not be restricted to the square-integrable solution of Eq. (18), we prefer to call it the Weber equation.

In order that $V_s(z_j)$ in Eq. (17) be well defined, we shall furthermore impose the restriction

$$q \geqslant -s \tag{18a}$$

Equation (18a) guarantees that the Weber functions present in Eq. (17) are strictly positive. This is explicitly apparent if we consider the expansion[8]

$$\begin{aligned} y_1\left(\frac{q}{2s},(2s)^{1/2}z\right) &= \exp\left(\frac{-sz^2}{2}\right) {}_1F_1\left(\frac{q}{4s}+\frac{1}{4},\frac{1}{2},sz^2\right) \\ &= \exp\left(\frac{-sz^2}{2}\right)\sum_{n=0}^{\infty}\frac{(q/4s+1/4)_n}{(1/2)_n n!}(sz^2)^n \end{aligned} \tag{18b}$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{18c}$$

and ${}_1F_1(a,b,z)$ stands for a Kummer (confluent hypergeometric) function.[8]

From the asymptotic expansion of the Kummer function,[9] we deduce[3]

$$\lim_{|z|\to\infty} V_s(z) \simeq \frac{(p-s\epsilon)}{2}z^2, \qquad \forall q > -s \tag{19}$$

Equation (19) leads us to impose

$$p - s\epsilon > 0 \tag{20}$$

which will guarantee the stability of our class of models.

[3] $q = -s$ is a singular point in the parameter space of the Kummer function and leads to a different behavior for $V_s(z)$; see Eq. (23b) below. This is the consequence of the singularity of the gamma function at the origin.

Using Eq. (18), and the parity of the $y_1(q(2s)^{-1},(2s)^{1/2}z)$ function, we find

$$\frac{d^2}{dz^2}V_s(z)\bigg|_{z=0}=p-q\epsilon \tag{21}$$

and hence,

$$V_s(z) \text{ is double well when } \quad p<q\epsilon \tag{22a}$$

$$V_s(z) \text{ is single well when}^4 \quad p>q\epsilon \tag{22b}$$

In words, the class of potentials Eq. (17) can be viewed as the perturbations of an harmonic oscillator, perturbations which may lead to single versus double-well structures.

For special values of the parameters, the potential Eq. (17) exhibits the following behaviors:

$$s=0\Rightarrow V_s(z)=\frac{p}{2}z^2-\epsilon\ln\left[\cosh\left(\sqrt{q}\,z\right)\right] \tag{23a}$$

$$s=-q\Rightarrow V_s(z)=\left(\frac{p+s\epsilon}{2}\right)z^2 \tag{23b}$$

$$q=0\Rightarrow V_s(z)=\frac{p}{2}z^2-\epsilon\ln\left[\left|\sqrt{s}\,z\right|^{1/2}I_{-\frac{1}{4}}\left(\frac{sz^2}{2}\right)\right] \tag{23c}$$

$$q=s\Rightarrow V_s(z)=\left(\frac{p-s\epsilon}{2}\right)z^2 \tag{23d}$$

$$q=(4n+1)s,\ n\in\mathbb{N}\backslash\{0\}$$

$$\Rightarrow V_s(z)=\frac{(p-s\epsilon)z^2}{2}-\epsilon\ln\left[\frac{(-1)^n n!}{(2n)!}H_{2n}\left(i\sqrt{s}\,z\right)\right] \tag{23e}$$

where in Eq. (23c), $I_{-\frac{1}{4}}(x)$ stands for a Bessel function of fractional order and in Eq. (23e) appear the Hermite polynomials of purely imaginary arguments. In Sections 4 and 5 we shall treat the special examples, Eqs. (23a) and (23e) with $n=1$.

For further convenience, we shall rescale the parameters in order that formally two of them are scaled away, namely, ϵ and s. This is achieved by using the following transformation in Eqs. (3), (7), (8), and (17):

$$t'=2sDt \tag{24a}$$

$$z'=(2s)^{1/2}z \tag{24b}$$

[4] The occurrence of more complex structures is ruled out by considering expansions of the type Eq. (18b) and remarking that with Eq. (18a), the Kummer's functions involved are strictly monotonous (see Ref. 10 for a more detailed consideration).

and hence,

$$p' = p(2s\epsilon)^{-1}, \qquad \theta' = \theta(2s\epsilon)^{-1}, \qquad D' = D(\epsilon)^{-1}$$
$$q' = q(2s)^{-1}, \qquad s' = \tfrac{1}{2}, \qquad \epsilon' = 1 \tag{24c}$$

In the rescaled parameters, the potential Eqs. (8) and (17) then reads (we omit the primes)

$$\begin{aligned} V_1(\mathbf{z}) &= \sum_{j=1}^{N}\left(\frac{p+\theta}{2}\right)z_j^2 - \sum_{j=1}^{N}\ln[y_1(q,z)] \\ &= \sum_{j=1}^{N}\left(\frac{p+\theta+\frac{1}{2}}{2}\right)z_j^2 - \sum_{j=1}^{N}\ln\left\{{}_1F_1\left(\frac{q}{2}+\frac{1}{4},\frac{1}{2},\frac{z_j^2}{2}\right)\right\} \end{aligned} \tag{25}$$

Let us remark at this stage that the case $s = 0$, Eq. (23a) is not covered by the scaling transformation Eq. (24c). In Sections 4 and 5, we shall deal with this limiting case separately by starting the calculations immediately with Eq. (23a).

Introducing Eq. (25) into Eq. (9a), we obtain

$$\begin{aligned} \exp[\psi(i\xi)] = \int_{\mathbb{R}} dz \exp\left[i\xi z - \left(\frac{p+\theta+\frac{1}{2}}{2D}\right)z^2\right] \\ \times\left[{}_1F_1\left(\frac{q}{2}+\frac{1}{4},\frac{1}{2},\frac{z^2}{2}\right)\right]^{D^{-1}} \end{aligned} \tag{26}$$

For the scaled diffusion constant $D = 1$, the integral Eq. (26) is exactly calculable (see Appendix A). For $D \neq 1$, Eq. (26) is only calculable in certain special situations to be considered in Section 5. Finally, for $D \simeq 1$, we introduce an approximation which reads

$$\exp[\psi(i\xi)] \cong \int_{\mathbb{R}} dz \exp\left(i\xi z - \frac{p+\theta}{2D}z^2\right)y_1\left(q,\frac{z}{\sqrt{D}}\right) \tag{27}$$

In other words, in Eq. (27) we have used the approximation

$$[y_1(q,z)]^{D^{-1}} \simeq y_1\left(q,\frac{z}{\sqrt{D}}\right) \tag{27a}$$

The approximation Eq. (27a) possesses the following properties:

$$\lim_{|z|\to\infty}[y_1(q,z)]^{D^{-1}} \simeq \lim_{|z|\to\infty} y_1\left(q,\frac{z}{\sqrt{D}}\right) \simeq \exp\left(\frac{z^2}{2D}\right) \tag{28a}$$

and

$$\left.\frac{d^2}{dz^2}[y_1(q,z)]^{D^{-1}}\right|_{z=0} = \left.\frac{d^2}{dz^2}y_1\left(q,\frac{z}{\sqrt{D}}\right)\right|_{z=0} = \frac{q}{D} \tag{28b}$$

Hence, from Eqs. (28a) and (28b), we observe that both the curvature at the origin and the leading behavior in the asymptotic region are preserved. While for the cases of scaled $q = \mp 1/2$ [note Eq. (24c); harmonic potential according to Eqs. (23b) and (23d)] Eq. (27a) is a strict equality, for D in the vicinity of the unity, the approximation is meaningful and will be observed to lead to consistent results.

Using Eq. (27), Eq. (26) then reads

$$\exp[\psi(i\xi)] = \int_{\mathbb{R}} dz \exp\left[i\xi z - \left(\frac{p+\theta+\frac{1}{2}}{2D} \right) z^2 \right] {}_1F_1\left(\frac{q}{2} + \frac{1}{4}, \frac{1}{2}, \frac{z^2}{2D} \right) \tag{29}$$

The integral, Eq. (29), is performed in the Appendix A. The result, Eq. (A.4), when substituted into Eq. (9) gives

$$P_s(x) = C \exp\left(\frac{N\theta x^2}{2D} \right) \int_{\mathbb{R}} d\xi \exp\left[-i\xi N x - \frac{N D \xi^2}{2(p+\theta+\frac{1}{2})} \right] \times \left[{}_1F_1\left[\frac{q}{2} + \frac{1}{4}, \frac{1}{2}, \frac{-D\xi^2}{2\left[(p+\theta)^2 - \frac{1}{4}\right]} \right] \right]^N \tag{30}$$

where C is a constant.

Finally, the integral in Eq. (30) is performed in Appendix B. The result given in Eq. (B.6), is in a closed operator form which reads [note Eq. (18b) for the series form of ${}_1F_1(a, b, z)$]

$$P_s(x) = \hat{Q} \exp\left(\frac{\theta N x^2}{2D} \right) \left[{}_1F_1\left(\frac{q}{2} + \frac{1}{4}, \frac{1}{2}, \frac{D}{2N^2\left[(p+\theta)^2 - \frac{1}{4}\right]} \frac{d^2}{dx^2} \right) \right]^N \times \exp\left[\frac{-N(\theta+p+\frac{1}{2})}{2D} x^2 \right] \tag{31}$$

where the normalization factor reads

$$\hat{Q} = \sqrt{N}\, Q^{-1} (2\pi D)^{(N-1)/2} \left(p+\theta+\tfrac{1}{2}\right)^{N(q/2-1/4)+1/2} \times \left(p+\theta-\tfrac{1}{2}\right)^{-N(q/2+1/4)} \tag{31a}$$

3. GENERAL DISCUSSION

On the basis of our result Eq. (31), let us make the following remarks:

(a) For $D = 1$, Eq. (31) is an exact expression valid for any number N of the members of the assembly. For D in the vicinity of unity, Eq. (31) is approximate with the only approximation being Eq. (27). Furthermore, we show in Appendix C that when $N = 1$, we indeed recover the single-particle result.

(b) When $q = \mp 1/2$ and $\forall D \in \mathbb{R}^+$, Eq. (31) reduces to the (exact) Gaussian results which are expected for the harmonic potentials Eqs. (23b) and (23d). (See Appendix B for an explicit calculation.)

(c) The operator form Eq. (31) should be understood as the formally exact (when $D = 1$) perturbation series around a Gaussian situation. As we have already remarked, the class of potentials, Eq. (17) and Eq. (18), can be viewed as a perturbation (not necessarily small) of an harmonic potential. Hence, Eq. (31) can itself be interpreted as the associated perturbations series around the Gaussian density expected for harmonic potentials.

(d) For large values of the argument $[(p + \theta + \frac{1}{2})D^{-1}]^{1/2}x$, the operator form Eq. (31) can be approximately expressed in a much simpler form. In Appendix D, we show that

$$\begin{aligned} P_s(x) \underset{\sqrt{\alpha}\,x \gg 1}{\simeq} \hat{Q} \exp\left\{ \frac{-N(p+\frac{1}{2})x^2}{2D} \right\} \\ \times \left[{}_1F_1\left(\frac{q}{2} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2D}\left(\frac{p+\theta+\frac{1}{2}}{p+\theta-\frac{1}{2}} \right) x^2 \right) \right]^N \end{aligned} \tag{32}$$

where

$$\alpha = 2(p + \theta + \tfrac{1}{2})D^{-1} \tag{32a}$$

In particular Eq. (32) shows that when the coupling parameter θ becomes very large, Eq. (32) presents a single oscillator behavior. Indeed we have

$$P_s(x) \underset{\sqrt{\alpha}\,x \gg 1}{\simeq} (N\theta)^{1/2} \left(\frac{2\pi D}{\theta} \right)^{N/2} Q^{-1} \exp\left[\frac{-NV_s(x)}{D} \right] \tag{33}$$

where $V_s(x)$ is the one-particle term of the potential introduced in Eq. (8). This behavior is intuitively clear, as for an infinite coupling the behavior of one member of the assembly totally influences the whole population.

Equation (32) can itself be further approximated for $x \gg 1$ by using the asymptotic expansion of the Kummer's function.[9] We have

$$\begin{aligned} P_s(x) \underset{\sqrt{\alpha}\,x \gg 1}{\simeq} \hat{Q} \exp\left\{ -(2D)^{-1} N \left[(p + \tfrac{1}{2}) + \left(\frac{p+\theta+\frac{1}{2}}{p+\theta-\frac{1}{2}} \right) \right] x^2 \right\} \\ \times \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{q}{2}+\frac{1}{4})} \left[(2D)^{-1} \left(\frac{p+\theta+\frac{1}{2}}{p+\theta-\frac{1}{2}} \right) x^2 \right]^{N(q/2-1/4)} \end{aligned} \tag{34}$$

The leading term in Eq. (34) is of Gaussian nature. This behavior is reminiscent of the fact that $V_1(z)$ is asymptotically harmonic [Eq. (19)].

Finally, let us remark that

$$\lim P_s(x) = 0, \qquad |x| \to \infty$$

which is required for the probability density to be normalizable.

(e) Having obtained the asymptotic expansion of Eq. (31), one would also like to study the behavior of Eq. (31) in the vicinity of the origin. This behavior is in principle obtainable by keeping only the constant and quadratic terms of the Hermite polynomials present in Eq. (B.5) of Appendix B. We are in general not able to resum the associated series in a compact form and this precludes our drawing the phase diagrams [obtained from the curvature of $P_s(x)$ at $x = 0$] in the general situation. This complexity can be traced back to the fact that the vicinity of the origin is precisely where the perturbation of the harmonic potential is the strongest.

In the thermodynamic limit, we may, however, use the Lemma introduced in Section 2 to obtain phase diagrams. Indeed, using Eq. (10) together with Eq. (A.4) of Appendix A, we obtain

$$R = \text{const}\left[\frac{-p-\frac{1}{2}}{D} + \frac{(p+\theta+\frac{1}{2})(q+\frac{1}{2})}{D(p+\theta+q)}\right] \tag{35}$$

From Eq. (35), we find the critical coupling θ_c given by the equation

$$R(\theta_c) = 0 \Rightarrow \theta_c = \frac{p^2 - \frac{1}{4}}{q - p} \tag{36}$$

Both Eqs. (35) and (36) are exact for $D = 1$ and $N = \infty$. From Eq. (35) we also observe that increasing the strength of the fluctuations (increasing D) flattens the curvature, which is physically reasonable and shows that our approximation Eq. (20) is consistent. Moreover, in the limit of large coupling $\theta \to \infty$, Eq. (35) reads

$$\lim_{\theta\to\infty} R = \frac{\text{const}}{D}[-p+q] \tag{37}$$

which is precisely the curvature obtained when only one member of the assembly is present.

Moreover from Eq. (36) we observe that

$$\theta_c \geqslant 0 \Rightarrow p \in [\tfrac{1}{2}, q] \tag{37a}$$

The boundary $\theta_c = 0$ is obtained when $p = 1/2$, which according to Eqs. (20) and (24c) corresponds to the limit of stability of our models. The other limit $\theta_c = \infty$ which is reached for $p = q$ corresponds to the transition between single- and double-well potential [see Eqs. (22a) and (22b)]. A typical phase diagram is sketched in Fig. 1.

Let us remark that Eq. (37a) clearly shows that a necessary condition to have an ordered phase $P_s(x)$ (bimodal) is that the single particle potential $V_s(z)$ Eq. (17), exhibits a double-well structure. This condition

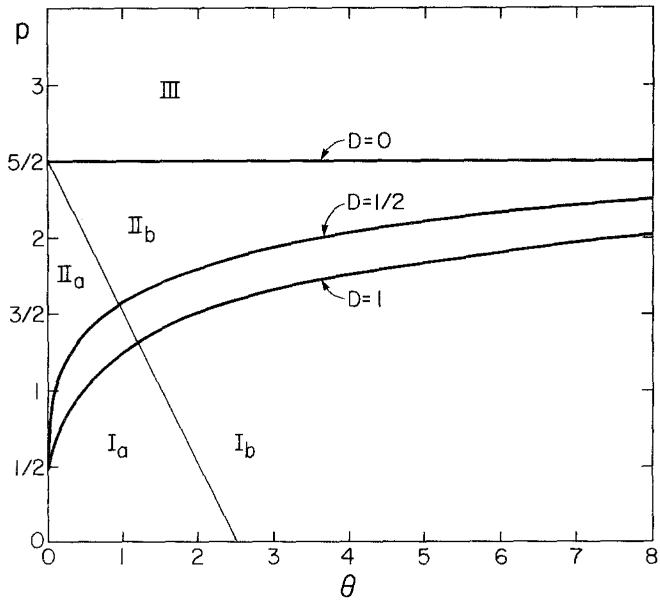

Fig. 1. Phase diagram for $q = 5/2$, $s = 1/2$, $D = 1, 1/2, 0$, $\epsilon = 1$, and $N \to \infty$. Region Ia: $P_s(x)$ bimodal, $V_1(z)$ double-well, order–disorder type. Region Ib: $P_s(x)$ bimodal, $V_1(z)$ single-well, displacive type. Region IIa: $P_s(x)$ unimodal, $V_1(z)$ double-well. Region IIb: $P_s(x)$ unimodal, $V_1(z)$ single-well. Region III: $P_s(x)$ unimodal, $V_s(z)$ single-well.

also holds for the class of models considered in Ref. 3. At this stage, we shall separate two types of transitions which do appear for the present class of models, namely, transitions of order–disorder type and of displacive type. The distinction is carried out according to whether the effective potential $V_1(z)$ in Eq. (8) is itself of single- or double-well structure. We shall speak of transition of order–disorder type when $V_1(z)$ is double well and of displacive type when $V_1(z)$ is single well. Using Eqs. (8) and (21), we have immediately

$$V_1(z) \text{ is single well when } p < (-\theta + q) \tag{38a}$$

$$V_1(z) \text{ is double well when } p > (-\theta + q) \tag{38b}$$

The separation line between the two regimes is sketched in Fig. 1.

4. SPECIAL CASES—SYSTEM SIZE EFFECTS

We shall now consider more closely two particular cases of our class of models for which the mathematics become particularly simple. While in this section we again restrict ourselves to $D \simeq 1$, Section 5 will deal with arbitrary strength of the fluctuations. In this section, we shall not, however, restrict ourselves to the thermodynamic limit and therefore we shall be able to discuss system site effects.

4.1. The Case $q = 5/2$ ($s = 1/2$)

For this value of the parameter q, the potential Eq. (25) simply reads [see Eq. (23e) with $n = 1$]:

$$V_1(\mathbf{z}) = \sum_{j=1}^{N} \left[\left(\frac{p + \theta - \frac{1}{2}}{2} \right) z_j^2 - \ln\left(1 + z_j^2\right) \right] \tag{39}$$

In Appendix E, we calculate the probability density for the order parameter when the approximation Eqs. (27) and (27a) hold. Using Eq. (E.2) we have

$$\begin{aligned} \underset{q=5/2}{P_s(x)} = \tilde{Q} \exp\left[\frac{-N(p - \frac{1}{2})x^2}{2D} \right] \\ \times \sum_{k=0}^{N} \binom{N}{k} \left(\frac{1}{N\tilde{\alpha}} \right)^k H_{2k} \left\{ \left[\frac{N(\tilde{\alpha} - 2)}{4D} \right]^{1/2} x \right\} \end{aligned} \tag{40}$$

where

$$\tilde{Q} = Q^{-1}\sqrt{N}\,(2\pi D)^{(N-1)/2}(p + \theta + \tfrac{1}{2})^N (p + \theta - \tfrac{1}{2})^{-3N/2 + 1/2} \tag{40a}$$

and

$$\tilde{\alpha} = 2(p + \theta + \tfrac{1}{2}) \tag{40b}$$

From Eq. (40), we calculate the curvature at the origin of $P_s(x)$. We have

$$R = N^{-1} \frac{d^2}{dx^2} \left(\left. \underset{q=5/2}{P_s(x)} \right|_{x=0} \right) = \frac{\tilde{Q}}{D} (-p + \tfrac{1}{2}) S_1(\tilde{\alpha}, N) + \frac{\tilde{Q}}{N} S_2(\tilde{\alpha}, N) \tag{41}$$

where

$$S_1(\tilde{\alpha}, N) = \sum_{k=0}^{N} \binom{N}{k} (N\tilde{\alpha})^{-k} (-1)^k (2k)! \,(k!)^{-1} \tag{41a}$$

and

$$\begin{aligned} S_2(\tilde{\alpha}, N) &= \frac{N(\tilde{\alpha} - 2)}{D} \sum_{k=1}^{N} \binom{N}{k} (N\tilde{\alpha})^{-k} (-1)^{k-1} (2k)! \left[(k-1)! \right]^{-1} \\ &= \frac{-N(\tilde{\alpha} - 2)}{\tilde{\alpha} D} \frac{d}{d(\tilde{\alpha}^{-1})} S_1(\tilde{\alpha}, N) \end{aligned} \tag{41b}$$

Using the identity[11] [see Eq. (18c) for the notation]

$$\binom{N}{k} = (-1)^k(-N)_k(k!)^{-1} \tag{42}$$

we obtain

$$S_1(\tilde{\alpha}, N) = {}_2F_0\left(-N, \tfrac{1}{2}, 4(N\tilde{\alpha})^{-1}\right) \tag{43}$$

where ${}_2F_0(a,b,x)$ stands for a generalized hypergeometric function.[12] Equations (41b) and (43) imply

$$R = QD^{-1}\Bigg[\left(-p+\tfrac{1}{2}\right){}_2F_0\left(-N, \tfrac{1}{2}, 4(N\tilde{\alpha})^{-1}\right) + 2\left(\frac{\tilde{\alpha}-2}{\tilde{\alpha}}\right){}_2F_0\left(-N+1, \tfrac{3}{2}, 4(N\tilde{\alpha})^{-1}\right)\Bigg] \tag{44}$$

where in Eq. (44) we have used the fact that

$$\frac{d}{dx}\,{}_2F_0(a,b,x) = ab\,{}_2F_0(a+1, b+1, x) \tag{44a}$$

Equation (44) clearly shows that the curvature R is system size dependent and that increasing D flattens the curvature. Moreover, in the strong coupling limit ($\theta \to \infty \Rightarrow \tilde{\alpha} \to \infty$), Eq. (44) simply reduces to

$$R = \tilde{Q}D^{-1}\left[-p+\tfrac{5}{2}\right] \tag{45}$$

where we have used the identity

$${}_2F_0(a,b,0) \equiv 1 \tag{46}$$

Equation (45) is immediately recognized to be the curvature one would obtain if only one member of the assembly were present (remember that in this case $q = 5/2$).

For large N, we may perform a system size expansion of Eq. (44). This is achieved by observing

$$\begin{aligned} S_1(\tilde{\alpha}, N) &= {}_2F_0\left(-N, \tfrac{1}{2}, 4(N\tilde{\alpha})^{-1}\right) \\ &\cong {}_1F_0\left(\tfrac{1}{2}, -4(\tilde{\alpha})^{-1}\right) \\ &\quad - 8(N\tilde{\alpha}^2)^{-1}\frac{d^2}{d\left[-4(\tilde{\alpha})^{-1}\right]^2}\,{}_1F_0\left(\tfrac{1}{2}, -4(\tilde{\alpha})^{-1}\right) + O(N^{-2}) \end{aligned} \tag{47}$$

Using the relation[13]

$${}_1F_0(a,x) = (1-x)^{-a}$$

Eqs. (47), (41a), and (41b) imply

$$R \cong \tilde{Q}D^{-1}\left(\frac{p+\theta+\frac{1}{2}}{p+\theta+\frac{5}{2}}\right)^{1/2}$$
$$\times\left\{-p-\tfrac{1}{2}+3\left(\frac{p+\theta+\frac{1}{2}}{p+\theta+\frac{5}{2}}\right)\right.$$
$$\left.+\left[\frac{15}{2N}\left(\frac{(p+\frac{1}{2})(p-\frac{1}{2})+\theta(\frac{9}{5}p+\frac{3}{5}\theta-\frac{1}{2})}{(p+\theta+\frac{5}{2})^3}\right)\right]\right\} \tag{48}$$

In the thermodynamic limit ($N \to \infty$), Eq. (48) immediately reduces to our previous result Eq. (35) (for $q = 5/2$) and hence leads to

$$\theta_c = \frac{p^2-\frac{1}{4}}{\frac{5}{2}-p} \tag{49}$$

Moreover, let us remark that according to Eqs. (20) and (24c), the square bracket in Eq. (48) is positive definite. This boundary term has then the tendency to increase the curvature and hence reduces the value of θ_c for a given p. From Eq. (48), we do observe that θ_c will be the solution of a cubic equation obtained from $R(\theta_c) = 0$. It is interesting to remark that in the limiting situation for which $p = 1/2$ and $\theta = 0$, the curvature R as given in Eq. (48) identically vanishes. This behavior is of course expected for the free particle model [$p = 1/2$ is the limit of stability: see Eq. (20), with the rescaled $s = 1/2$].

4.2. The Case $s = 0$

As mentioned before, this situation should be considered separately as the scaling transformation, Eqs. (24a)–(24c), becomes meaningless in this limit. Hence we reconsider Eq. (23a) and the density for the order parameter reads [see Eq. (9)]

$$\underset{s=0}{P_s(x)} = (2\pi Q)^{-1}N\exp\left\{\frac{\theta N x^2}{2\hat{D}}\right\}\int_{\mathbb{R}} d\xi \exp\left[-i\xi N x + N\psi(i\xi)\right] \tag{50}$$

where

$$\hat{D} = D/\epsilon \tag{50a}$$

and

$$\exp\left[\psi(i\xi)\right] = \int_{\mathbb{R}} dz \exp\left[i\xi z - \frac{(p+\theta)}{2\hat{D}}z^2\right]\left[\cosh\left(\sqrt{q}\,z\right)\right]^{\hat{D}^{-1}} \tag{51}$$

For $\hat{D}$ in the vicinity of unity, we again use the approximation Eq. (27) which leads us to write Eq. (51) in the form

$$\exp[\psi(i\xi)] \cong \int_{\mathbb{R}} dz \exp\left[i\xi z - \frac{(p+\theta)}{2\hat{D}} z^2\right] \cosh\left[\left(\frac{q}{\hat{D}}\right)^{1/2} z\right] \tag{51a}$$

Using Eq. (51a), the integral in Eq. (50) is performed in Appendix F, where Eq. (F.3) leads to

$$P_s(x) = \bar{Q} \exp\left(\frac{-Npx^2}{2\hat{D}}\right) \sum_{k=0}^{(N-1)/2} \binom{N}{k} 2^{-N} \times \exp\left[\frac{q(N-2k)^2}{2(p+\theta)N}\right] \cosh\left[\left(\frac{q}{\hat{D}}\right)^{1/2} (N-2k)x\right] \tag{52}$$

where $\bar{Q}$ is given in Appendix F.

At this stage, let us remark that Eq. (52) is valid for $N = 2n+1$, $n \in N$. A similar calculation for N even, can also be performed but we do not consider it here. In particular in the thermodynamic limit $N \to \infty$ both N odd or even lead to the same results which we have explicitly verified.

Using Eq. (52), the curvature R of $P_s(x)$ at the origin reads

$$R = N^{-1} \frac{d^2}{dx^2} P_s(x)\bigg|_{x=0} = \bar{Q}\hat{D}^{-1}\left[-p\hat{S}_1(\hat{\alpha}, N) + N^{-1}\hat{S}_2(\hat{\alpha}, N)\right] \tag{53}$$

where

$$\hat{\alpha} = \tfrac{1}{2}(p+\theta) \tag{53a}$$

$$\hat{S}_1(\hat{\alpha}, N) = \sum_{k=0}^{(N-1)/2} \binom{N}{k} 2^{-N} \exp\left[\frac{-q(N-2k)^2}{4N\hat{\alpha}}\right] \tag{53b}$$

and

$$\hat{S}_2(\hat{\alpha}, N) = q \sum_{k=0}^{(N-1)/2} \binom{N}{k} 2^{-N} \exp\left[\frac{-q(N-2k)^2}{4N\hat{\alpha}}\right](N-2k)^2 = -qN \frac{d}{d\left[q(4\tilde{\alpha})^{-1}\right]} \hat{S}_1(\hat{\alpha}, N) \tag{53c}$$

Using Eqs. (53a)–(53c), the curvature R in Eq. (53) is again observed to be system size dependent.

In the thermodynamic limit, the sum $\hat{S}_1(\hat{\alpha}, N)$ can be calculated in the continuous limit and by using the property that

$$\lim_{N\to\infty} 2^{-N}\binom{N}{k} = \frac{2}{(2\pi N)^{1/2}} \exp\left[-\left(\frac{N}{2} - k\right)^2 N^{-1}\right] \tag{54}$$

With Eq. (54), we have

$$\lim_{N\to\infty} \hat{S}_1(\hat{\alpha}, N) = \int_{\mathbb{R}^+} dk \frac{2}{(2\pi N)^{1/2}} \exp\left[\frac{-(N/2-k)^2}{N}\left(2+\frac{q}{2}\right)\right]$$
$$= \left(\frac{2\hat{\alpha}}{2\hat{\alpha}+q}\right)^{1/2} \tag{55}$$

Using Eq. (55) and (53c), the curvature Eq. (53) reduces to

$$R = \overline{Q}\hat{D}^{-1}\left(\frac{p+\theta}{p+\theta+q}\right)^{1/2}\left[-p+q\left(\frac{p+\theta}{p+q+\theta}\right)\right] \tag{56}$$

and hence

$$R(\theta_c) = 0 \Rightarrow \theta_c = \frac{p^2}{q-p} \tag{57}$$

From Eq. (57) we can draw the phase diagram—Fig. 2.

Moreover, Eq. (56) indicates that, in the limit of validity of Eq. (27), to increase the noise strength D has again the effect of flattening R. Furthermore, for $\theta \to \infty$, the single oscillator behavior is also immediately recovered, namely,

$$\underset{\theta\to\infty}{R} = \hat{Q}\hat{D}^{-1}(-p+q) \tag{58}$$

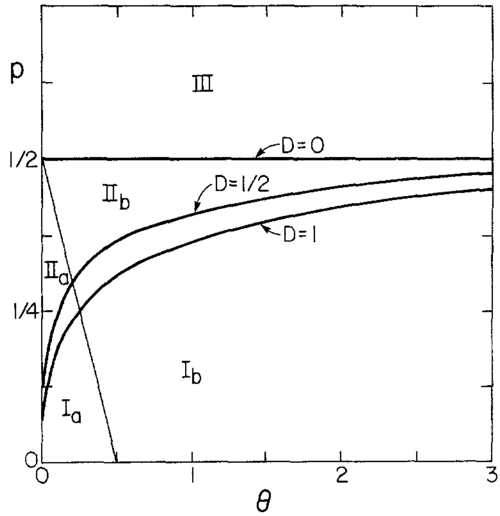

Fig. 2. Phase diagram for $q = 1/2$, $s = 0$, $D = 1, 1/2$, $\epsilon = 1$, and $N \to \infty$. Regions as in Fig. 1.

Let us close this section by noting that Eq. (57) can also be directly obtained from our previous result Eq. (36) in the limit $p, q \to \infty$. This limit is indeed reached when $s \to 0$ in our scaling transformation Eq. (24c).

5. PHASE DIAGRAM FOR ARBITRARY NOISE INTENSITY

In this section, we shall study the behavior of our models for arbitrary noise intensity D. We shall again consider the special cases $q = 5/2$ and $s = 0$ and restrict ourselves to the thermodynamic limit $N \to \infty$ for which our Lemma Eq. (10) holds.

Let us start with the case $q = 5/2$ which by using Eqs. (9a) and (39) leads us to

$$\exp[\psi(\omega)] = \int_{\mathbb{R}} \exp(\omega z - \zeta z^2)(1 + z^2)^F dz$$
$$= \left(\frac{\pi}{\zeta}\right)^{1/2} \exp\left(\frac{\omega^2}{4\zeta}\right) \sum_{\lambda=0}^{\infty} \binom{F}{\lambda} \left(\frac{-1}{4\zeta}\right)^F H_{2\lambda}\left(\frac{i\omega}{2\sqrt{\zeta}}\right) \tag{59}$$

where

$$F = D^{-1} \tag{59a}$$

$$\zeta = (p + \theta - \tfrac{1}{2})(2D)^{-1} \tag{59b}$$

$$\omega = i\xi \qquad (i = \sqrt{-1}) \tag{59c}$$

Now we will use our Lemma Eq. (10) and therefore, we have to calculate

$$B_0 = \frac{d^2}{d\omega^2} \psi(\omega)\bigg|_{\omega=0} = \frac{1}{2\zeta} + \frac{d^2}{d\omega^2} \ln \sum_{\lambda=0}^{\infty} \binom{F}{\lambda} \left(\frac{-1}{4\zeta}\right)^{\lambda} H_{2\lambda}\left(\frac{i\omega}{2\sqrt{\zeta}}\right)\bigg|_{\omega=0}$$
$$= \frac{1}{2\zeta} + \frac{1}{\zeta^2} \frac{d}{d(1/\zeta)} \ln {}_2F_0\left(-F, \frac{1}{2}, -\frac{1}{\zeta}\right) \tag{60}$$

Then using Eqs. (10) and (60), we immediately have

$$R = \text{const}\left\{ \frac{\theta}{D} - \left[\frac{1}{2\zeta} + \frac{1}{\zeta^2} \frac{d}{d(1/\zeta)} \ln {}_2F_0\left(-F, \frac{1}{2}, \frac{-1}{\zeta}\right)\right]^{-1} \right\} \tag{61}$$

Equation (61) provides us with the exact curvature R of $P_s(x)$ for arbitrary $D = F^{-1}$, $q = 5/2$, and $N \to \infty$. The critical coupling θ_c, obtained from Eq. (61) by the equation $R(\theta_c) = 0$, is in general impossible to find by analytical means. Indeed, the equation $R(\theta_c) = 0$ is in general transcendental; it reduces to a polynomial of degree F when $F = D^{-1} \in N$. In the following we shall discuss three particular situations for which $R(\theta_c)$ takes however simpler forms. These situations are reached when $D = 1,\ 1/2,\ 0$.

5.1. $D = 1$

This case has already been considered in Eqs. (35) and $q = 5/2$ and in Eq. (48) with $q = 5/2$ and $N \to \infty$. We rederive it here just for the sake of completeness. We have

$$ {}_2F_0\left(-1, \frac{1}{2}, \frac{-1}{\zeta}\right) = 1 + \frac{1}{(2\zeta)} \tag{62} $$

and hence using Eq. (61) we obtain

$$ R = \text{const}\left[\theta - \frac{2\zeta(\zeta + \frac{1}{2})}{\zeta + \frac{3}{2}}\right] = \text{const}\left[-p - \frac{1}{2} + 3\left(\frac{p + \theta + \frac{1}{2}}{p + \theta + \frac{5}{2}}\right)\right] \tag{63} $$

5.2. $D = 1/2$

In this case Eq. (60) takes the form:

$$ \begin{aligned} B_0 &= \frac{y}{2} + y^2 \frac{d}{dy}\left[\ln {}_2F_0\left(-2, \frac{1}{2}, -y\right)\right] \\ &= \left(\tfrac{15}{4} y^3 + 3y^2 + y\right)\left(2 + 2y + \tfrac{3}{2} y^2\right)^{-1} \end{aligned} \tag{64} $$

where in Eq. (64), we have used the fact that

$$ {}_2F_0(-2, \tfrac{1}{2}, -y) = 1 + y + \tfrac{3}{4} y^2 \tag{65} $$

and the notation $y = \zeta^{-1}$.

Using Eq. (64), the curvature Eq. (60) reads

$$ R = \text{const}\left[-2p + 1 + 2\left(\frac{2y^{-2} + 3y^{-1}}{15/4 + 3y^{-1} + y^{-2}}\right)\right] \tag{66} $$

and hence

$R(\theta_c) = 0 \Rightarrow$

$$ \theta_c = \frac{2(p-2)(p+\frac{1}{2}) + \left[4(2-p)^2(p+\frac{1}{2}) + (5-2p)(p-\frac{1}{2})(2p^2+1)\right]^{1/2}}{5 - 2p} \tag{67} $$

Equation (67) leads to the phase diagram sketched in Fig. 1 for $D = 1/2$. Figure 1 explicitly shows that decreasing the strength of the fluctuations causes the system to approach the behavior of the single oscillator, a result in agreement with our intuition.

5.3. $D = 0$

Let us finally consider the case of vanishingly small fluctuations. In this case, we have

$$\lim_{D^{-1}=F\to\infty} {}_2F_0\left(-F, \frac{1}{2}, -\frac{1}{\hat{\xi}F}\right) = \left(1 - \frac{1}{\hat{\xi}}\right)^{-1/2} + O(F^{-1}) \tag{68}$$

where

$$\hat{\xi} = D\xi \tag{68a}$$

Using Eq. (69), Eq. (61) takes the form

$$B_0 = \frac{D}{2\hat{\xi}} + \frac{D}{\hat{\xi}^2}\frac{d}{d(1/\hat{\xi})}\ln\left[\left(1 - \frac{1}{\hat{\xi}}\right)^{-1/2}\right] = \frac{D}{2}\left[\frac{1}{\hat{\xi}-1}\right] \tag{69}$$

Remark that Eq. (69) explicitly indicates that we have kept the first order in D.

Using Eq. (69) into Eq. (10), we have

$$\lim_{D\to 0} R = \frac{\text{const}}{D}\left[\theta - 2(\hat{\xi} - 1)\right] = \frac{\text{const}}{D}\left[-p + \tfrac{5}{2}\right] \tag{70}$$

Equation (70) is independent of θ and hence the phase diagram is degenerate. Equation (70) gives the same curvature as the one that would have been obtained if only one member of the assembly were present. This clearly indicates that we are in the deterministic limit and hence the phase diagram is degenerate with a step function: Fig. 1 ($D = 0$).

Let us finally investigate the case $s = 0$ in the thermodynamic limit $N \to \infty$ and for arbitrary noise strength.

Using the same technique as above we have with Eq. (23a)

$$\begin{aligned}\exp[\psi(\omega)] &= \int_{\mathbb{R}} \exp\left[\omega z - \left(\frac{p+\theta}{2}\right)\hat{F}z^2\right]\left[\cosh\left(\sqrt{q}\,z\right)\right]^{\hat{F}} dz \\ &= \left[\frac{2\pi}{\hat{F}(p+\theta)}\right]^{1/2}\exp\left[\frac{\omega^2}{2(p+\theta)\hat{F}}\right] \\ &\quad\times \sum_{k=0}^{\infty}\binom{\hat{F}}{k}2^{-\hat{F}}\exp\left[\frac{q(\hat{F}-2k)^2}{2\hat{F}(p+\theta)}\right]\cosh\left[\frac{\omega\sqrt{q}\,(\hat{F}-2k)}{\hat{F}(p+\theta)}\right]\end{aligned} \tag{71}$$

where

$$\hat{F} = \frac{\epsilon}{D} = \hat{D}^{-1} \tag{71a}$$

From Eq. (71), we obtain

$$\left.\frac{d^2}{d\omega^2}\psi(\omega)\right|_{\omega=0} = \frac{1}{(p+\theta)\hat{F}} + \frac{2q}{\hat{F}(p+\theta)}\frac{d}{dq}\ln\left[\Sigma_1(p,\theta,\hat{F})\right] \tag{72}$$

where

$$\Sigma_1(p,\theta,\hat{F}) = \sum_{k=0}^{\infty} 2^{-\hat{F}}\binom{\hat{F}}{k}\exp\left[\frac{q(\hat{F}-2k)^2}{2\hat{F}(p+\theta)}\right] \tag{72a}$$

In particular for $\hat{F} = 1 = \epsilon/D$, we have

$$\left.\frac{d^2}{d\omega^2}\psi(\omega)\right|_{\omega=0} = \frac{p+q+\theta}{(p+\theta)^2} \tag{73}$$

which immediately leads to [see Eq. (10)]

$$R = \text{const}\left[-p + q\left(\frac{p+\theta}{p+\theta+q}\right)\right] \tag{74}$$

Equation (74) is identical with our previous result Eq. (56).

For $\hat{F} = 2$, we obtain from Eqs. (72) and (72a)

$$\left.\frac{d^2}{d\omega^2}\psi(\omega)\right|_{\omega=0} = \frac{1}{2(p+\theta)} + \frac{q}{(p+\theta)^2\chi(p,\theta)} \tag{75}$$

where

$$\chi(p,\theta) = 1 + \exp\left[-q(p+\theta)^{-1}\right] \in [1,2] \tag{76}$$

Using Eq. (75) and Eq. (10) we find

$$R = \text{const}\,2\left[-p + q\left[\frac{2(p+\theta)}{p+\theta+2q+(p+\theta)\exp\left[-q(p+\theta)^{-1}\right]}\right]\right] \tag{77}$$

From Eq. (77), the critical coupling is implicitly given by the transcendental equation

$$R(\theta_c) = 0 \tag{78}$$

When the ratio $q(p+\theta)^{-1} \ll 1$, we may expand the exponential which occurs in Eq. (77) to get

$$\underset{q(p+\theta)^{-1}\ll 1}{R} \cong \text{const}\,2\left[-p + \frac{2q(p+\theta)}{2p+2\theta+q}\right] \tag{79}$$

and hence in this limit we obtain

$$R(\theta_c) = 0 \Rightarrow \theta_c \simeq \frac{2p^2 - pq}{q - p} \tag{80}$$

In the region where $p < q$ (namely, where the ordered phase can appear), we observe that when $q/(p + \theta) \ll 1$

$$\underset{D=1/2}{\theta_c} \simeq \frac{2p^2 - pq}{q - p} < \frac{p^2}{q - p} = \underset{D=1}{\theta_c} \tag{81}$$

Equation (81) indicates once again that decreasing the fluctuations strength decreases the critical coupling θ_c, a result in agreement with our previous observation.

In Figure 2 we sketch the phase diagram for the case $s = 0$ and $D = 1$, $1/2$ by using Eqs. (57) and (78).

6. CONCLUSIONS AND SUMMARY

We have studied the equilibrium properties of a class of models which consists of an assembly of coupled nonlinear oscillators in the Schmoluchowski limit. The coupling is of system size range and each member of the assembly is subject to a single versus double-well type potential according to a set of external parameters. In our class of models, the oscillators are statistically independent and hence the stationary probability density can be found immediately (detailed balance) and permits us to express the probability density of an order parameter defined as the arithmetic mean of oscillator-coordinates.

While for quartic type potential, a similar study has been performed in Ref. 3, we consider here perturbed harmonic potential which in contrast to Ref. 3 permits us to obtain exact analytical results. Indeed, the perturbed harmonic potentials we introduce here are expressible in terms of Weber parabolic cylinder functions which possess the important property of having self-similar Fourier transforms (a precious property shared by Gaussians). The main results of our paper can be summarized as follows:

(1) For a special value of the noise intensity D (namely, unity in our choice of scaled parameters), we obtain in the thermodynamic limit exact phase diagrams for the whole class of models. Our results follow from a lemma (introduced in Section 2) which is a straightforward consequence of the steepest descent method, and which can be used to write the marginal probability density of the order parameter in the $N \to \infty$ limit. For D in the vicinity of unity, we provide an approximation Eq. (27) which is found to lead to consistent results.

(2) In the phase diagram obtained, we distinguish between two types of phase transitions which occur for the class of models. Namely, we have transitions of the order–disorder type which occur when the effective potential, V_1 in Eq. (8), is of double-well type and of displacive type which arise when the effective potential is itself a single well.

For $D = 1$ and special choices of the external parameter, which control the shape of the potential, we can solve the problem exactly for any number of oscillators. This allows us in particular to perform a system size expansion to obtain the correction term due to the finite size of the system (boundary term). See Eq. (48). This boundary term is observed to have a definite sign. Moreover, for the same intensity of the noise and identical values of the parameters, we stress that a finite size system may exhibit a disordered behavior while the same system considered in the thermodynamic limit presents an overall order. This is a concrete illustration of the remark made in Ref. 1 which shows that the strength of the noise in cooperative systems can be altered by changing the number of the members composing the assembly. This fact then strongly suggests that one should take special care when using the steepest descent method, especially near the transition points.

(3) Finally, for special values of the parameters, we are able to obtain exact results for arbitrary noise intensities when the thermodynamic limit is considered. The dependence of the phase diagram on the noise intensity D is shown in Figs. 1 and 2. In the limit of vanishingly small fluctuations, we are also able to observe the way the phase boundary between disordered and ordered regions degenerates into the expected step function.

APPENDIX A

Let us consider the integrals of the form

$$\exp[\psi(i\xi)] = \int_{\mathbb{R}} dz \exp(i\xi z - \alpha z^2)\, {}_1F_1(a, \tfrac{1}{2}, \beta z^2) \tag{A.1}$$

We first introduce the integral representation[14]

$${}_1F_1(a, \tfrac{1}{2}, \beta z^2) = \frac{2}{\Gamma(a)} \int_{\mathbb{R}^+} d\lambda \exp(-\lambda^2)\lambda^{2a-1}\cosh(2\sqrt{\beta}\,\lambda z) \tag{A.2}$$

Introducing Eq. (A.2) into Eq. (A.1), reversing the order of the integrations, and performing one integration gives us

$$\exp[\psi(i\xi)] = \left(\frac{\pi}{\alpha}\right)^{1/2} \frac{2}{\Gamma(a)} \exp[-\xi^2(4\alpha)^{-1}] \times \int_{\mathbb{R}^+} d\lambda \exp[-\lambda^2(1-\beta/\alpha)]\lambda^{2a-1}\cos(\sqrt{\beta}\,\alpha^{-1}\xi\lambda) \tag{A.3}$$

The integral in Eq. (A.3) is a variant of Eq. (A.1) and can be found in Ref. 15 with the result

$$\exp\left[\psi(i\xi)\right] = \sqrt{\pi}\,\alpha^{a-1/2}(\alpha-\beta)^{-a}\exp\left[-\xi^2(4\alpha)^{-1}\right]$$
$$\times\, {}_1F_1\left(a, \frac{1}{2}, -\frac{\beta\xi^2}{4\alpha(\alpha-\beta)}\right) \tag{A.4}$$

Then using the notation

$$\alpha = (p+\theta+\tfrac{1}{2})(2D)^{-1}$$
$$\beta = (2D)^{-1}$$

and $a = q/2 + 1/4$, we find Eq. (30).

APPENDIX B

Here, we consider the integrals of the type

$$I_q(x) = \int_{\mathbb{R}} d\xi \exp\left(i\xi N x - \frac{N\xi^2}{\alpha}\right)\left[{}_1F_1\left(a, \tfrac{1}{2}, -\beta\xi^2\right)\right]^N \tag{B.1}$$

Let us introduce the notation

$${}_1F_1\left(a, \tfrac{1}{2}, -\beta\xi^2\right) = \lim_{M\to\infty}\sum_{k=0}^{M} A_k\left(-\beta\xi^2\right)^k \tag{B.2}$$

where

$$A_k = (k!)^{-1}(a)_k\left[\left(\tfrac{1}{2}\right)_k\right]^{-1} \tag{B.2a}$$

Further, we introduce the multinomial formula

$$(x_1 + x_2 + \cdots x_p)^N = \tilde{\sum} \frac{N!}{n_1!\,n_2!\ldots n_p!}(x_1)^{n_1}(x_2)^{n_2}\ldots(x_p)^{n_p} \tag{B.3}$$

where the sum $\tilde{\sum}$ is taken over all nonnegative integers n_j ($j = 1 \ldots p$) for which we have $\sum_{j=1}^{p} n_j = N$.

Hence, Eqs. (B.3) and (B.2) imply

$$\left[{}_1F_1\left(a, \tfrac{1}{2}, -\beta\xi^2\right)\right]^N = \lim_{M\to\infty}\tilde{\sum}\phi(M, n_k)\left(-\beta\xi^2\right)^{\theta(k)} \tag{B.4}$$

where

$$\phi(M, n_k) = N!\,\frac{\prod_{k=0}^{M}(A_k)^{n_k}}{\prod_{k=0}^{M}(n_k)!} \tag{B.4a}$$

and

$$\theta(k) = \sum_{k=0}^{M} k n_k \tag{B.4b}$$

Introducing Eq. (B.4) into Eq. (B.1) and performing the integration gives

$$I_q(x) = \left(\frac{\pi\alpha}{N}\right)^{1/2} \exp\left(-\frac{N\alpha x^2}{4}\right) \times \lim_{M\to\infty} \widetilde{\sum} \phi(M, n_k)\left(\frac{\beta\alpha}{4N}\right)^{\theta(k)} H_{2\theta(k)}\left(\left(\frac{N\alpha}{4}\right)^{1/2} x\right) \tag{B.5}$$

Finally, using the very definition of the Hermite polynomials,[16] Eq. (B.5) can be written in the form

$$\begin{aligned} I_q(x) &= \left(\frac{\pi\alpha}{N}\right)^{1/2}\left[\lim_{M\to\infty} \widetilde{\sum} \phi(M, n_k)\left(\frac{\beta}{N^2}\right)^{\theta(k)} \frac{d^{2\theta(k)}}{dx^{2\theta(k)}} \exp\left(-\frac{N\alpha x^2}{4}\right)\right] \\ &= \left(\frac{\pi\alpha}{N}\right)^{1/2}\left[{}_1F_1\left(a, \frac{1}{2}, \frac{\beta}{N^2} \frac{d^2}{dx^2}\right)\right]^N \exp\left(-\frac{N\alpha x^2}{4}\right) \end{aligned} \tag{B.6}$$

Then using the notations

$$\beta = \frac{D}{2\left[(p+\theta)^2 - 1/4\right]} = \frac{2}{\alpha(\alpha D - 2)} \tag{B.6a}$$

$$a = q/2 + 1/4 \tag{B.6b}$$

and

$$\alpha = 2D^{-1}\left(p + \theta + \tfrac{1}{2}\right) \tag{B.6c}$$

Eq. (31) is obtained.

Let us check the validity of Eq. (B.6) for the special case when the rescaled $q = 1/2$, which according to Eq. (23d) should reduce to the Gaussian results. Indeed, we have

$$\begin{aligned} I_{1/2}(x) &= \left(\frac{\pi\alpha}{N}\right)^{1/2}\left[\exp\left(\frac{\beta}{N^2}\frac{d^2}{dx^2}\right)\right]^N \exp\left(-\frac{N\alpha x^2}{4}\right) \\ &= \left(\frac{\pi\alpha}{N}\right)^{1/2} \exp\left(\frac{-N\alpha x^2}{4}\right) \sum_{k=0}^{\infty}\left(\frac{\beta\alpha}{4}\right)^k \frac{1}{k!} H_{2k}\left[\frac{(N\alpha)^{1/2} x}{2}\right] \\ &= \left(\frac{\pi\alpha}{N}\right)^{1/2}\left[\sum_{k=0}^{\infty}(-\beta\alpha)^k \frac{(1/2)_k}{k!}\, {}_1F_1\left(-k, \frac{1}{2}, \frac{N\alpha x^2}{4}\right)\right] \end{aligned} \tag{B.7}$$

At this stage, we introduce the multiplication theorem[17]

$$ {}_1F_1(a,b,yz) = y^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(y-1)^n}{(y)^n n!}\,{}_1F_1(a+n,b,z) \tag{B.8} $$

Then using the Kummer transform in Eq. (B.7) and the expansion Eq. (B.8), we obtain

$$ I_{1/2}(x) = \left[\frac{\pi\alpha}{N(1+\alpha\beta)}\right]^{1/2}\exp\left[-\frac{N\alpha x^2}{4(1+\alpha\beta)}\right] \tag{B.9} $$

which is indeed of the expected form.

APPENDIX C

For $N = 1$ Eq. (31) reads

$$ \begin{aligned} P_s(x) &= \hat{Q}\exp\left(\frac{\theta x^2}{2D}\right)\sum_{n=0}^{\infty}\frac{(a)_n(\beta)^n}{(1/2)_n n!}\frac{d^{2n}}{dx^{2n}}\exp\left(\frac{-N\alpha x^2}{4}\right) \\ &= \hat{Q}\exp\left(\frac{-\alpha x^2}{4}+\frac{\theta}{2D}x^2\right)\sum_{n=0}^{\infty}\frac{(a)_n(\beta\alpha)^n}{(1/2)_n n!\,4^n}H_{2n}\left[\left(\frac{\alpha}{4}\right)^{1/2}x\right] \end{aligned} \tag{C.1} $$

At this stage, we use the generating function[18]

$$ \begin{aligned} (1-t)^{-p}\,{}_1F_1\left(p,\frac{1}{2},\frac{-zt}{1-t}\right) &= \sum_{n=0}^{\infty}\frac{(p)_n L_n^{-1/2}(z)t^n}{(1/2)_n} \\ &= \sum_{n=0}^{\infty}\frac{(p)_n(-1)^n H_{2n}(\sqrt{z})t^n}{(1/2)_n 4^n n!} \end{aligned} \tag{C.2} $$

where $L_n^{\alpha}(z)$ stands for a generalized Laguerre polynomial. With Eq. (C.2), Eq. (C.1) becomes

$$ \begin{aligned} P_s(x) &= \hat{Q}\left(\frac{\alpha D-2}{\alpha D}\right)^{-a}\exp\left[\frac{-(p+\frac{1}{2})x^2}{2D}\right]{}_1F_1\left(a,\frac{1}{2},\frac{x^2}{2D}\right) \\ &= Q^{-1}\exp\left[\frac{-(p+\frac{1}{2})}{2D}x^2\right]{}_1F_1\left(\frac{q}{2}+\frac{1}{4},\frac{1}{2},\frac{x^2}{2D}\right) \end{aligned} \tag{C.3} $$

which is indeed the single-particle result in the approximation Eq. (27) for $D \simeq 1$.

APPENDIX D

Let us now consider the large $\sqrt{\alpha}\,x$ asymptotic behavior of Eq. (31); α is defined by Eq. (B.6c). To do this, we keep the highest degree of the Hermite polynomials in the expansion Eq. (B.5). We then have

$$\underset{\sqrt{\alpha}\,x\gg 1}{I_q(x) \simeq} \left(\frac{\pi\alpha}{N}\right)^{1/2} \exp\left(-\frac{N\alpha x^2}{4}\right) \lim_{M\to\infty}\left[\widetilde{\sum}\phi(M,n_k)\left(\frac{\beta\alpha}{4N}\right)^{\theta(k)}(N\alpha x^2)^{\theta(k)}\right]$$

$$= \left(\frac{\pi\alpha}{N}\right)^{1/2} \exp\left(\frac{-N\alpha x^2}{4}\right)\left[{}_1F_1\left(a,\frac{1}{2},\frac{\beta\alpha^2 x^2}{4}\right)\right]^N \tag{D.1}$$

where the notations are given in Eqs. (B.6) and (B.6a)–(B.6c).

APPENDIX E

When $q = 5/2$, Eq. (B.1) takes the particular form

$$I_{5/2}(x) = \int_{\mathbb{R}} d\xi \exp\left(-i\xi N x - \frac{N\xi^2}{\alpha}\right)\left[{}_1F_1\left(\tfrac{3}{2},\tfrac{1}{2},\beta\xi^2\right)\right]^N$$

$$= \int_{\mathbb{R}} d\xi \exp\left[-i\xi N x - N\xi^2\left(\frac{1+\alpha\beta}{\alpha}\right)\right](1-2\beta\xi^2)^N$$

$$= \sum_{k=0}^{N}\binom{N}{k}(-2\beta)^k \exp\left[\frac{-N\alpha x^2}{4(1+\alpha\beta)}\right]$$

$$\times \int_{\mathbb{R}} d\xi\, \xi^{2k} \exp\left\{-\left[\sqrt{N}\,\xi\left(\frac{1+\alpha\beta}{\alpha}\right)^{1/2} - \frac{ix(\alpha N)^{1/2}}{2(1+\alpha\beta)^{1/2}}\right]^2\right\} \tag{E.1}$$

The integral Eq. (E.1) is given in Ref. 19 and we obtain

$$I_{5/2}(x) = \left[\frac{\pi\alpha}{N(1+\alpha\beta)}\right]^{1/2} \exp\left[\frac{-N\alpha x^2}{4(1+\alpha\beta)}\right]$$

$$\times \sum_{k=0}^{N}\binom{N}{k}\left[\frac{\beta\alpha}{2N(1+\alpha\beta)}\right]^k H_{2k}\left[\left(\frac{N\alpha}{1+\alpha\beta}\right)^{1/2}\frac{x}{2}\right] \tag{E.2}$$

where the notations are given in Eq. (B.6) and in particular

$$\alpha(1+\alpha\beta)^{-1} = (p+\theta-\tfrac{1}{2})(2D)^{-1} \tag{E.2a}$$

APPENDIX F

Here we consider integrals of the type

$$\underset{s=0}{P_s(x)} = N(2\pi Q)^{-1}\exp\left(\frac{\theta N x^2}{2D}\right)\int_{\mathbb{R}} d\xi \exp\left[-i\xi N x + N\psi(i\xi)\right] \quad \text{(F.1)}$$

where

$$\exp\left[\psi(i\xi)\right] = \int_{\mathbb{R}} dz \exp(i\xi z - \hat{\alpha} z^2)\cosh \hat{\beta} z \quad \text{(F.1a)}$$

$$\hat{\alpha} = (p+\theta)(2D)^{-1} \quad \text{(F.1b)}$$

$$\hat{\beta} = \left(\frac{q}{D}\right)^{1/2} \quad \text{(F.1c)}$$

After performing the integration over z Eq. (F.1) becomes

$$\underset{s=0}{P_s(x)} = N(2\pi Q)^{-1}\left(\frac{\pi}{\alpha}\right)^{N/2}\exp\left(\frac{N\theta x^2}{2D} + \frac{N\hat{\beta}^2}{4\hat{\alpha}}\right)$$

$$\times \int_{\mathbb{R}} d\xi \exp\left(-i\xi N x - \frac{N\xi^2}{4\hat{\alpha}}\right)\left[\cos\left(\frac{\xi\hat{\beta}}{2\hat{\alpha}}\right)\right]^N \quad \text{(F.2)}$$

For $N = 2K+1$, $K \in N$, we have[20]

$$\underset{s=0}{P_s(x)} = N(2\pi Q)^{-1}\left(\frac{\pi}{\alpha}\right)^{N/2}\exp\left\{\frac{\theta N x^2}{2D} + \frac{N\hat{\beta}^2}{4\hat{\alpha}}\right\}\frac{1}{2}\sum_{k=0}^{(N-1)/2}\binom{N}{k}2^{-N}$$

$$\times \int_{\mathbb{R}} d\xi \exp\left\{-i\xi N x - \frac{N\xi^2}{4\hat{\alpha}}\right\}\cos\left[(N-2k)\frac{\xi\hat{\beta}}{2\hat{\alpha}}\right]$$

$$= \bar{Q}\exp\left[-Nx^2\left(\hat{\alpha} - \frac{\theta}{2D}\right)\right]$$

$$\times \sum_{k=0}^{(N-1)/2}\binom{N}{k}2^{-N}\exp\left[-\frac{\hat{\beta}^2(N-2k)^2}{4\hat{\alpha}N}\right]\cosh\left(x\hat{\beta}(N-2k)\right)$$

(F.3)

where

$$\bar{Q} = \frac{1}{2}\sqrt{N}\left(Q\sqrt{\pi}\right)^{-1}\left(\frac{\pi}{\alpha}\right)^{N/2}\exp\left(\frac{N\hat{\beta}^2}{4\hat{\alpha}}\right) \quad \text{(F.3a)}$$

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